Introduction to Kinematics

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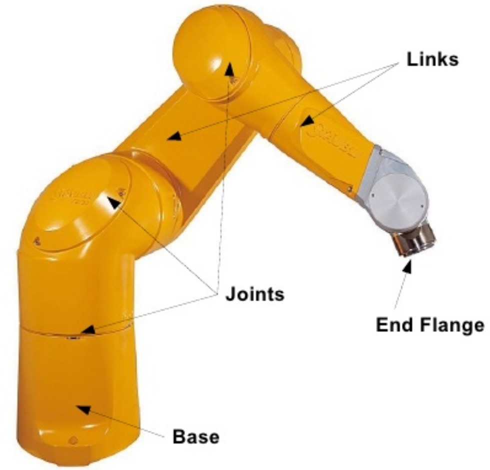
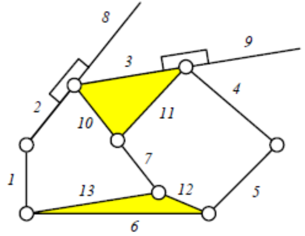
Theory of Applied Robotics , 2017

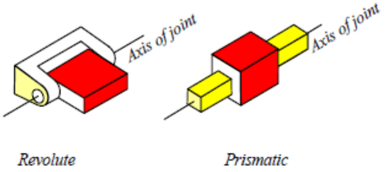
Outline

1. **Robot Components**
2. **Rotation kinematics**
   - Single Rotation
   - Successive Rotation
   - Aviation perspective
   - Examples
3. **Advanced issues**

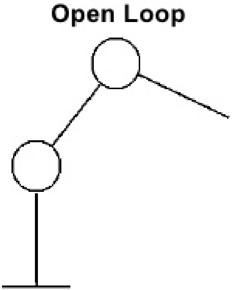
links and joints

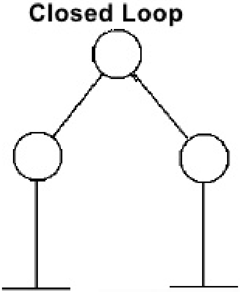






Closed loop Vs. Open loop





Z-Rotation Matrix About Global Cartesian Axes

$\theta = \alpha + \phi \qquad r_1 = r_2$

$X_1 = r_1 cos\phi \qquad Y_1 = r_1 sin\phi$

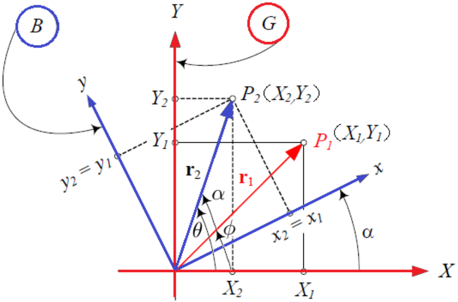
$X_2 = r_1 cos\theta \qquad Y_2 = r_1 sin\theta$

$X_2 = r_1 cos(\alpha + \phi) \quad Y_2 = r_1 sin(\alpha + \phi)$

$X_2 = r_1 cos(\alpha)cos(\phi) - r_1 sin(\alpha)sin(\phi)$

$X_2 = cos(\alpha)X_1 - sin(\alpha)Y_1$

$Y_2 = sin(\alpha)X_1 + cos(\alpha)Y_1$



$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

★ Rules

$cos(\alpha + \phi) = cos(\alpha)cos(\phi) - sin(\alpha)sin(\phi)$
$sin(\alpha + \phi) = sin(\alpha)cos(\phi) + cos(\alpha)sin(\phi)$

Z-Rotation Matrix About Local Cartesian Axes

$$\phi = \theta - \alpha \qquad r_1 = r_2$$

$$X_1 = r_1 cos\phi \qquad Y_1 = r_1 sin\phi$$

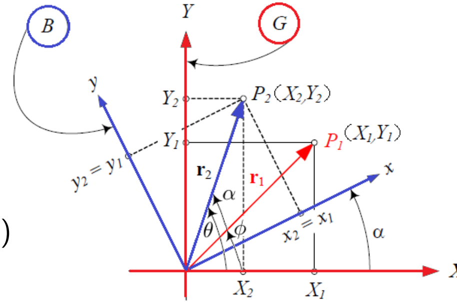
$$X_2 = r_1 cos\theta \qquad Y_2 = r_1 sin\theta$$

$$X_1 = r_1 cos(\theta - \alpha) \quad Y_1 = r_1 sin(\theta - \alpha)$$

$$X_1 = r_1 cos(\theta)cos(-\alpha) - r_1 sin(\theta)sin(-\alpha)$$

$$X_1 = cos(\alpha)X_2 + sin(\alpha)Y_2$$

$$Y_2 = -sin(\alpha)X_1 + cos(\alpha)Y_1$$



$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}$$

★ Rules

$$cos(\alpha + \phi) = cos(\alpha)cos(\phi) - sin(\alpha)sin(\phi)$$
$$sin(\alpha + \phi) = sin(\alpha)cos(\phi) + cos(\alpha)sin(\phi)$$

Z-Rotation Matrix About Local Cartesian Axes (In other words)

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}^{-1} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}^{-1}$$

$$= \frac{1}{c^2(\alpha) + s^2(\alpha)} \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix}$$

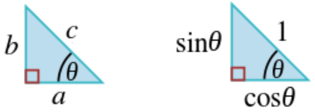
$$= \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} = {G_{R_L}}^T$$

$${G_{R_L}}^{-1} = {G_{R_L}}^T$$ Orthogonal matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

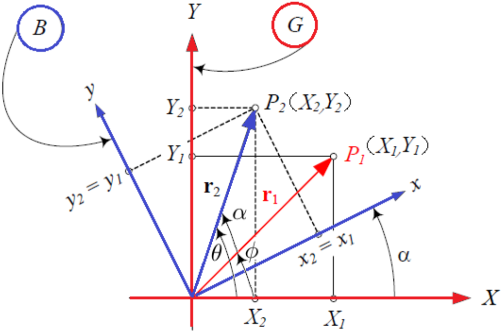
determinant (ad-bc)

Pythagorean Trigonometric



$$\sin^2(\theta) + \cos^2(\theta) = 1$$

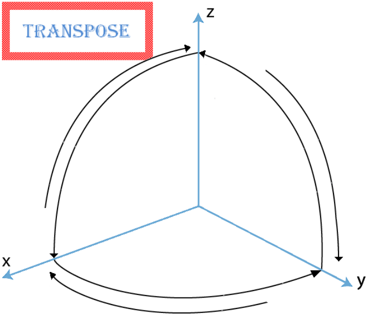
Z-Rotation Matrix About Global Cartesian/Local Cartesian



$$\textit{Global}: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \downarrow$$

$$\textit{Local}: \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \uparrow$$

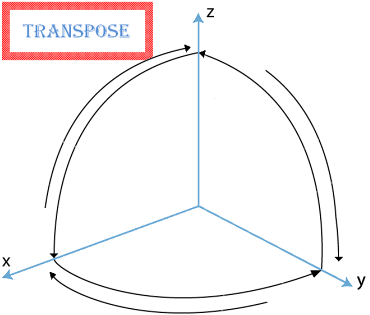
X-Rotation matrix About Global Cartesian/Local Cartesian



$$Global: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\gamma) & -sin(\gamma) \\ 0 & sin(\gamma) & cos(\gamma) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \downarrow$$

$$Local: \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\gamma) & sin(\gamma) \\ 0 & -sin(\gamma) & cos(\gamma) \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \uparrow$$

Y-Rotation matrix About Global Cartesian/Local Cartesian



$$Global: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \downarrow$$

$$Local: \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \uparrow$$

Twelve independent triple global rotations !!!

Circulate rotation:-

1. $Q_{X,\gamma} Q_{Y,\beta} Q_{Z,\alpha}$
2. $Q_{Y,\gamma} Q_{Z,\beta} Q_{X,\alpha}$
3. $Q_{Z,\gamma} Q_{X,\beta} Q_{Y,\alpha}$
4. $Q_{Z,\gamma} Q_{Y,\beta} Q_{X,\alpha}$
5. $Q_{Y,\gamma} Q_{X,\beta} Q_{Z,\alpha}$
6. $Q_{X,\gamma} Q_{Z,\beta} Q_{Y,\alpha}$

Double rotation:-

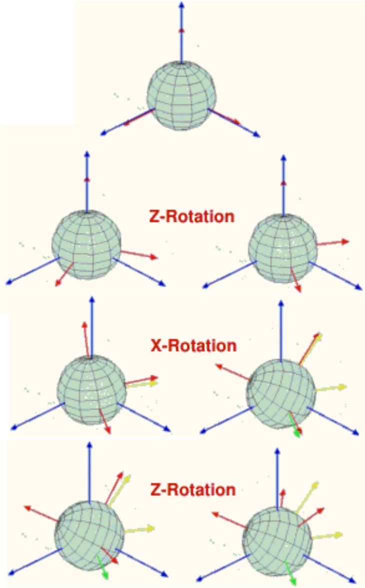
7. $Q_{X,\gamma} Q_{Y,\beta} Q_{X,\alpha}$
8. $Q_{Y,\gamma} Q_{Z,\beta} Q_{Y,\alpha}$
9. $Q_{Z,\gamma} Q_{X,\beta} Q_{Z,\alpha}$
10. $Q_{X,\gamma} Q_{Z,\beta} Q_{X,\alpha}$
11. $Q_{Y,\gamma} Q_{X,\beta} Q_{Y,\alpha}$
12. $Q_{Z,\gamma} Q_{Y,\beta} Q_{Z,\alpha}$

Order of rotation, IS order of matrix multiplication

The particular angle sequence is often a convention within a particular technological field.

what is the rotation sequences?! (Euler angles)

Leonhard Euler
1707 – 1783



what is the rotation sequences?! (Euler angles)

Leonhard Euler

1707 – 1783

proper/classic Euler angles;
precession-nutation-spin

7. $Q_{X,\gamma} Q_{Y,\beta} Q_{X,\alpha}$
8. $Q_{Y,\gamma} Q_{Z,\beta} Q_{Y,\alpha}$
9. $\boxed{Q_{Z,\gamma} Q_{X,\beta} Q_{Z,\alpha}}$
10. $Q_{X,\gamma} Q_{Z,\beta} Q_{X,\alpha}$
11. $Q_{Y,\gamma} Q_{X,\beta} Q_{Y,\alpha}$
12. $Q_{Z,\gamma} Q_{Y,\beta} Q_{Z,\alpha}$

Precession-Nutation-Spin

$Q_{Z,\varphi} Q_{X,\theta} Q_{z,\psi}$

General Rule

Always we offer Global to Local, transpose if otherwise.

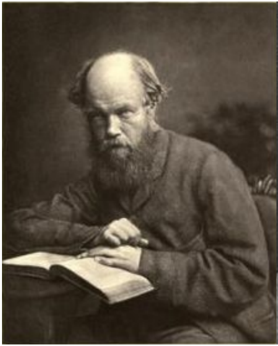
Global = Local

$$= \begin{bmatrix} cos(\varphi) & sin(\varphi) & 0 \\ -sin(\varphi) & cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & sin(\theta) \\ 0 & -sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

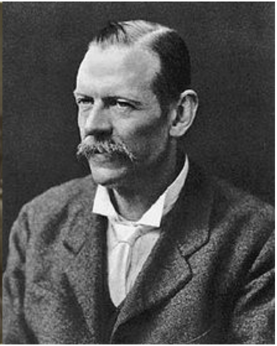
$$= \begin{bmatrix} c(\varphi) & c(\theta)s(\varphi) & s(\theta)s(\varphi) \\ -s(\varphi) & c(\theta)c(\varphi) & c(\varphi)s(\theta) \\ 0 & -s(\theta) & c(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c(\varphi)c(\psi) - c(\theta)s(\varphi)s(\psi) & c(\psi)s(\varphi) + c(\theta)c(\varphi)s(\psi) & s(\theta)s(\psi) \\ -c(\varphi)s(\psi) - c(\theta)c(\psi)s(\varphi) & -s(\varphi)s(\psi) + c(\theta)c(\varphi)c(\psi) & s(\theta)c(\psi) \\ s(\theta)s(\varphi) & -c(\varphi)s(\theta) & c(\theta) \end{bmatrix}$$

what is the rotation sequences?! (Tait-Bryan angles)



Peter Guthrie Tait
1831 – 1901
Scottish



George Hartley Bryan
1864 – 1928
Cambridge

Cardan angles;
Nautical angles;
Heading, Elevation, Bank;
Yaw, Pitch, Roll:-

1. $Q_{X,\gamma} Q_{Y,\beta} Q_{Z,\alpha}$
2. $Q_{Y,\gamma} Q_{Z,\beta} Q_{X,\alpha}$
3. $Q_{Z,\gamma} Q_{X,\beta} Q_{Y,\alpha}$
4. $Q_{Z,\gamma} Q_{Y,\beta} Q_{X,\alpha}$
5. $Q_{Y,\gamma} Q_{X,\beta} Q_{Z,\alpha}$
6. $Q_{X,\gamma} Q_{Z,\beta} Q_{Y,\alpha}$

Successive rotation Example

End effector = $\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} \rightarrow 30^\circ$ @ Z-axis $\rightarrow 30^\circ$ @ X-axis $\rightarrow 90^\circ$ @ Y-axis.

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} cos(30) & -sin(30) & 0 \\ sin(30) & cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(30) & -sin(30) \\ 0 & sin(30) & cos(30) \end{bmatrix} \begin{bmatrix} -10.68 \\ 28.48 \\ 10.0 \end{bmatrix} = \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix}$$

$$= \begin{bmatrix} cos(90) & 0 & sin(90) \\ 0 & 1 & 0 \\ -sin(90) & 0 & cos(90) \end{bmatrix} \begin{bmatrix} -10.68 \\ 19.66 \\ 22.9 \end{bmatrix} = \begin{bmatrix} 22.90 \\ 19.66 \\ 10.68 \end{bmatrix}$$

Zx-Rotation matrix

$$^{G}Q_B = Q_{X,\gamma} Q_{Y,\beta} Q_{Z,\alpha}$$

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} cos(\alpha) & -sin(\alpha) & 0 \\ sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\gamma) & -sin(\gamma) \\ 0 & sin(\gamma) & cos(\gamma) \end{bmatrix} \begin{bmatrix} c(\alpha)c(\beta) & -c(\beta)s(\alpha) & s(\beta) \\ s(\alpha) & c(\alpha) & 0 \\ -c(\alpha)s(\beta) & s(\alpha)s(\beta) & c(\beta) \end{bmatrix}$$

$$= \begin{bmatrix} c(\alpha)c(\beta) & -c(\beta)s(\alpha) & s(\beta) \\ c(\gamma)s(\alpha) + c(\alpha)s(\beta)s(\gamma) & c(\alpha)c(\gamma) - s(\alpha)s(\beta)s(\gamma) & -c(\beta)s(\gamma \\ s(\alpha)s(\gamma) - c(\alpha)c(\gamma)s(\beta) & c(\alpha)s(\gamma) + c(\gamma)s(\alpha)s(\beta) & c(\beta)c(\gamma) \end{bmatrix}$$

Global roll, pitch, and yaw rotations, respectivly

It is important to note that $R(\alpha, \beta, \gamma)$ performs the roll first, then the pitch, and finally the yaw. If the order of these operations is changed, a different rotation matrix would result. ace a 3D body in any orientation. A single rotation matrix can be formed by multiplying the yaw, pitch, and roll rotation matrices to obtain

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_x(\gamma)$$

$$\begin{bmatrix} c(\alpha)c(\beta) & -c(\beta)s(\alpha) & s(\beta) \\ c(\gamma)s(\alpha) + c(\alpha)s(\beta)s(\gamma) & c(\alpha)c(\gamma) - s(\alpha)s(\beta)s(\gamma) & -c(\beta)s(\gamma) \\ s(\alpha)s(\gamma) - c(\alpha)c(\gamma)s(\beta) & c(\alpha)s(\gamma) + c(\gamma)s(\alpha)s(\beta) & c(\beta)c(\gamma) \end{bmatrix}$$

Example: Back to origin

End effector = $\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix}$ of a rigid body is rotated successively about x-axis as follows:$120^\circ, 120^\circ, 120^\circ$, find the new position.

$$= \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix}^3_{\beta=120} = \begin{bmatrix} -0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & -0.5 \end{bmatrix}^3$$

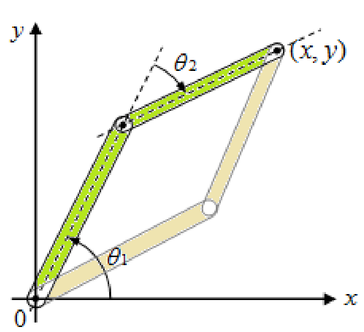
$$= \begin{bmatrix} -0.5 & 0 & -0.866 \\ 0 & 1 & 0 \\ 0.866 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} -0.5 & 0 & 0.866 \\ 0 & 1 & 0 \\ -0.866 & 0 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The end effector will return to the original position

(review: Example 10 pp 43)

Yes, all roads lead to Rome

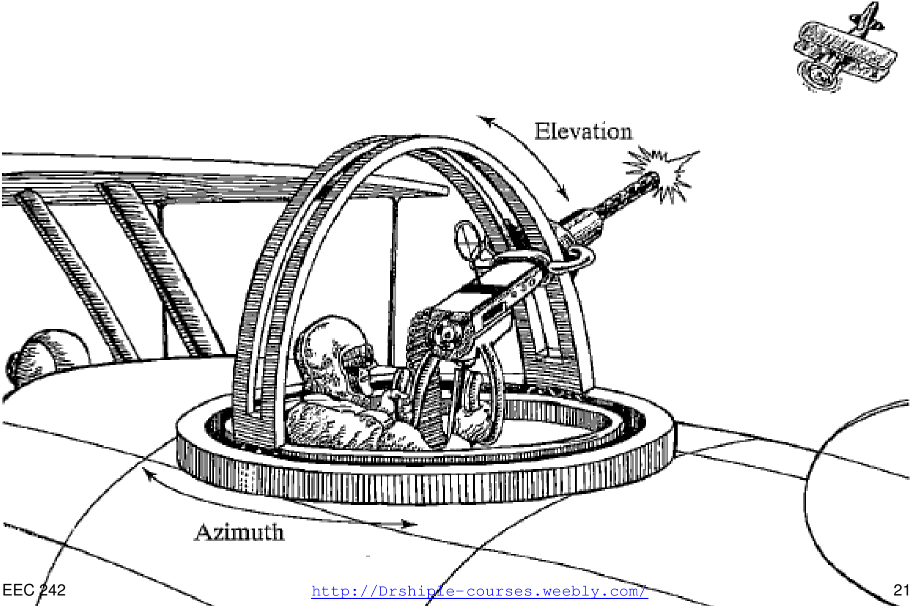


$$R(\varphi,\theta,\psi) = R_z(\psi)R_x(\theta)R_z(\varphi) = R_z(-40)R_x(41.41)R_z(79.15)$$

$$= \begin{bmatrix} 0.63 & 0.65 & -0.43 \\ -0.43 & 0.75 & 0.5 \\ 0.65 & -0.125 & 0.75 \end{bmatrix} = R_y(30)R_x(30)R_z(30)$$

Remark: to find from where you come, use certain angle procedure

Singularities and Gimbal Lock



Singularities and Gimbal Lock

$$= \begin{bmatrix} c(\varphi)c(\psi) - c(\theta)s(\varphi)s(\psi) & c(\psi)s(\varphi) + c(\theta)c(\varphi)s(\psi) & s(\theta)s(\psi) \\ -c(\varphi)s(\psi) - c(\theta)c(\psi)s(\varphi) & -s(\varphi)s(\psi) + c(\theta)c(\varphi)c(\psi) & s(\theta)c(\psi) \\ s(\theta)s(\varphi) & -c(\varphi)s(\theta) & c(\theta) \end{bmatrix}$$

assume $\theta = 0$ **Gimbal angle**

$$= \begin{bmatrix} c(\varphi)c(\psi) - s(\varphi)s(\psi) & c(\psi)s(\varphi) + c(\varphi)s(\psi) & 0 \\ -c(\varphi)s(\psi) - c(\psi)s(\varphi) & -s(\varphi)s(\psi) + c(\varphi)c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c(\varphi + \psi) & s(\varphi + \psi) & 0 \\ -s(\varphi + \psi) & c(\varphi + \psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the angles φ and ψ are indistinguishable even if the value of φ and ψ are finite.

Singularities and Gimbal Lock

$$= \begin{bmatrix} c(\varphi)c(\psi) - c(\theta)s(\varphi)s(\psi) & c(\psi)s(\varphi) + c(\theta)c(\varphi)s(\psi) & s(\theta)s(\psi) \\ -c(\varphi)s(\psi) - c(\theta)c(\psi)s(\varphi) & -s(\varphi)s(\psi) + c(\theta)c(\varphi)c(\psi) & s(\theta)c(\psi) \\ s(\theta)s(\varphi) & -c(\varphi)s(\theta) & c(\theta) \end{bmatrix}$$

assume $\theta, \psi, \varphi \approx 0$ very small degrees **Gimbal angle**

$$= \begin{bmatrix} 1 - s(\varphi)s(\psi) & s(\varphi) + s(\psi) & s(\theta)s(\psi) \\ -s(\psi) - s(\varphi) & -s(\varphi)s(\psi) + 1 & s(\theta) \\ s(\theta)s(\psi) & -s(\theta) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (\varphi + \psi) & 0 \\ -(\varphi + \psi) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the angles φ and ψ are indistinguishable