Orientation Kinematics

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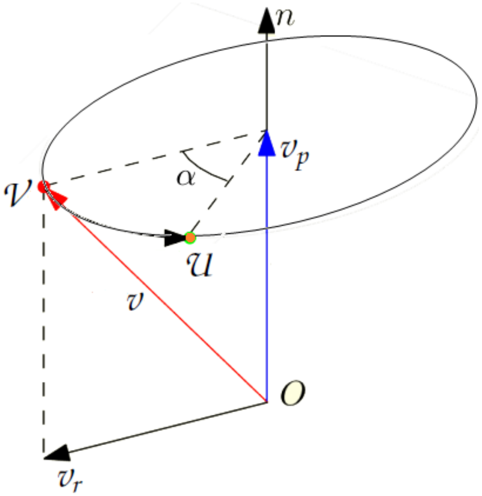
Theory of Applied Robotics , 2017

Outline

1. **Rodrigues' Formula**

Axis-angle Rotation



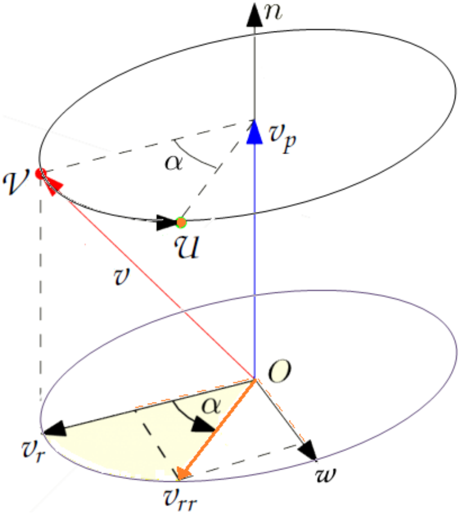
Benjamin Olinde Rodrigues
French banker, (1795 –1851)

$$v_p = (n \cdot v)n$$

$$v = v_r + v_p$$

$$v_r = v - v_p$$

Cont.:Axis-angle Rotation



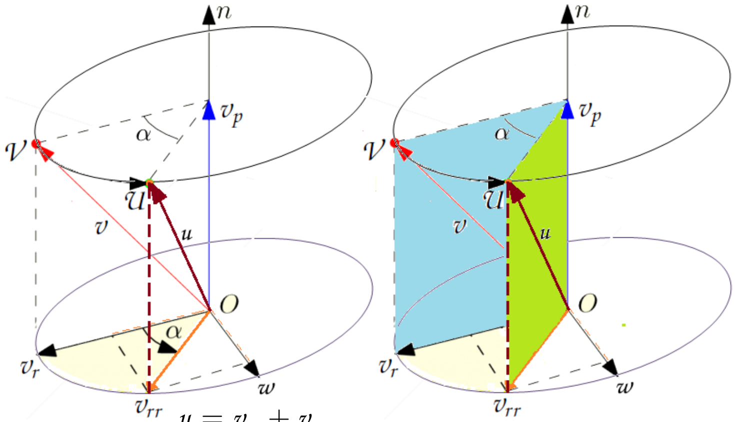
$$v_r = v_{rr} = v - v_p$$

$$v_r = v - (n \cdot v)n$$

$$v_{rr} = v_r \cos\alpha + w \sin\alpha$$

$$v_{rr} = (v - (n \cdot v)n)\cos\alpha + (n \times v)\sin\alpha$$

Cont.:Axis-angle Rotation



$$u = v_{rr} + v_p$$

$$v_{rr} = (v - (n \cdot v)n)\cos\alpha + (n \times v)\sin\alpha$$

$$u = (v - (n \cdot v)n)\cos\alpha + (n \times v)\sin\alpha + (n \cdot v)n$$

$$u = v\cos\alpha - (n \cdot v)n\cos\alpha + (n \times v)\sin\alpha + (n \cdot v)n$$

$$u = v\cos\alpha + (n \cdot v)n(1 - \cos\alpha) + (n \times v)\sin\alpha$$

Cont.:Axis-angle Rotation

$$u = v\cos\alpha + (n\cdot v)n(1-\cos\alpha) + (n\times v)\sin\alpha$$

$$v_p = (v\cdot n)n = (n^{\mathcal{T}}v)n = nn^{\mathcal{T}}v$$

$$\because \qquad n\times v = [n]_\times v$$

$$u = v\cos\alpha + nn^{\mathcal{T}}v(1-\cos\alpha) + [n]_\times v\sin\alpha$$

$$u = (I\cos\alpha + nn^{\mathcal{T}}(1-\cos\alpha) + [n]_\times\sin\alpha)v$$

Rodrigues' Formula describes the rotation of vector v around vector n with angle α so we can rewrite the formula as:

$$G_{R_B} = (I\cos\alpha + nn^{\mathcal{T}}(1-\cos\alpha) + [n]_\times\sin\alpha)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\cos\alpha + \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_1 n_3 & n_3^2 \end{pmatrix}(1-\cos\alpha)$$

$$+ \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}\sin\alpha$$

Cont.:Axis-angle Rotation

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} c\alpha + \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_1 n_3 & n_3^2 \end{pmatrix} (1 - c\alpha) + \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

assume the rotation vector n is the z axis: n=[0 0 1]

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} c\alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (1 - c\alpha) + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} s\alpha$$

$$= \begin{pmatrix} c\alpha & 0 & 0 \\ 0 & c\alpha & 0 \\ 0 & 0 & c\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1 - c\alpha) \end{pmatrix} + \begin{pmatrix} 0 & -s\alpha & 0 \\ s\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ref

- Rodrigues' rotation formula, http://electroncastle.com/wp/?p=39
- Mechanics of Robotic Manipulation, Matthew T. Mason