Butterworth Filter

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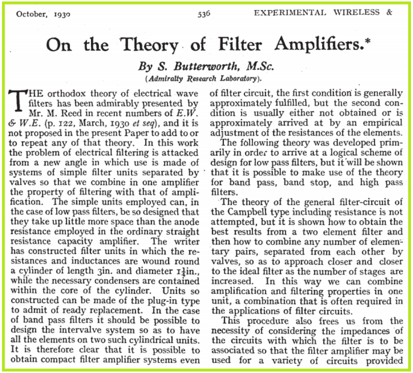
Shaping Circuits (EEC 242), 2015

Outline

Stephen Butterworth

- Stephen Butterworth (1885 – 1958) was a British physicist who invented the Butterworth filter.
- In 1939, he was a "Principal Scientific Officer" at the Admiralty Research Laboratory in the Admiralty's Scientific Research and Experiment Department.



October, 1930 536 EXPERIMENTAL WIRELESS &

On the Theory of Filter Amplifiers.*

By S. Butterworth, M.Sc.
(*Admiralty Research Laboratory*).

THE orthodox theory of electrical wave filters has been admirably presented by Mr. M. Reed in recent numbers of *E.W. & W.E.* (p. 122, March, 1930 *et seq*), and it is not proposed in the present Paper to add to or to repeat any of that theory. In this work the problem of electrical filtering is attacked from a new angle in which use is made of systems of simple filter units separated by valves so that we combine in one amplifier the property of filtering with that of amplification. The simple units employed can, in the case of low pass filters, be so designed that they take up little more space than the anode resistance employed in the ordinary straight resistance capacity amplifier. The writer has constructed filter units in which the resistances and inductances are wound round a cylinder of length 3in. and diameter 1¼in., while the necessary condensers are contained within the core of the cylinder. Units so constructed can be made of the plug-in type to admit of ready replacement. In the case of band pass filters it should be possible to design the intervalve system so as to have all the elements on two such cylindrical units. It is therefore clear that it is possible to obtain compact filter amplifier systems even of filter circuit, the first condition is generally approximately fulfilled, but the second condition is usually either not obtained or is approximately arrived at by an empirical adjustment of the resistances of the elements.

The following theory was developed primarily in order to arrive at a logical scheme of design for low pass filters, but it will be shown that it is possible to make use of the theory for band pass, band stop, and high pass filters.

The theory of the general filter-circuit of the Campbell type including resistance is not attempted, but it is shown how to obtain the best results from a two element filter and then how to combine any number of elementary pairs, separated from each other by valves, so as to approach closer and closer to the ideal filter as the number of stages are increased. In this way we can combine amplification and filtering properties in one unit, a combination that is often required in the applications of filter circuits.

This procedure also frees us from the necessity of considering the impedances of the circuits with which the filter is to be associated so that the filter amplifier may be used for a variety of circuits provided

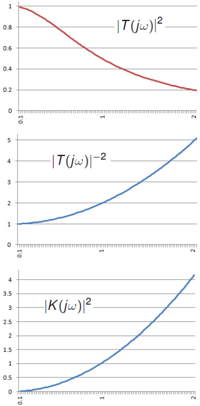
|T(S)| Equation

1. The selected equation should be conjugate: that means

$$|T(j\omega)| = Re\ T(j\omega) + j\ Im\ T(j\omega)$$
$$|T(-j\omega)| = Re\ T(j\omega) - j\ Im\ T(j\omega)$$
$$= |T^*(j\omega)|$$

define Attenuation Function

$$|K(j\omega)|^2 = |T(j\omega)|^{-2} - 1$$
$$|T(j\omega)|^2 = \frac{1}{1 + B_2\omega^2 + B_4\omega^4 + \cdots + B_{2n}\omega^{2n}}$$



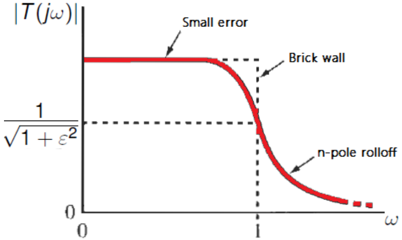
cont.|k(S)| Equation

$$|K(j\omega)|^2 = B_2\omega^2 + B_4\omega^4 + \cdots + B_{2n}\omega^{2n}$$ For max. flat

$$\left.\frac{d^k(|K(j\omega)|^2)}{d(\omega^2)^k}\right|_{\omega=0} = 0$$ For $k = 1, 2, \ldots, n-1$

$$|K(j\omega)|^2 = B_{2n}\omega^{2n} = \varepsilon^{2n}\omega^{2n}$$ $\varepsilon = 1$ for butterworth filter

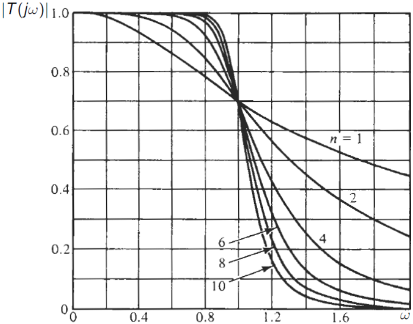
$$|T(j\omega)|^2 = \frac{1}{1+\varepsilon^{2n}\omega^{2n}} = \frac{1}{1+\omega^{2n}}$$



Butterworth properties

$$|T(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

- The Butterworth filter is all pole filter.
- $|T(j0)| = 1$.
- $|T(j1)| = 0.707 = \frac{1}{\sqrt{2}}$ for $\forall n$.
- The attenuation increases by 20n dB/decade.



Odd / Even number

$$|T(j\omega)| = \frac{1}{1 + (-1)^n S^{2n}}$$

$n = 1$

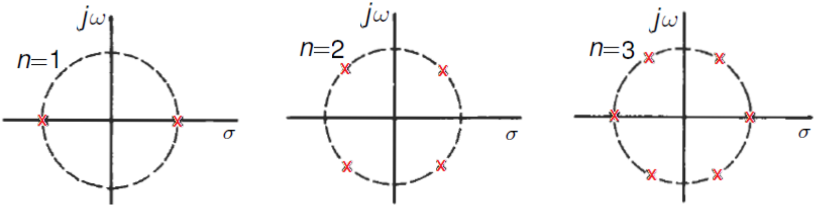
$$1 - S^2 = 0$$
$$(1 + S)(1 - S) = 0$$

$n = 2$

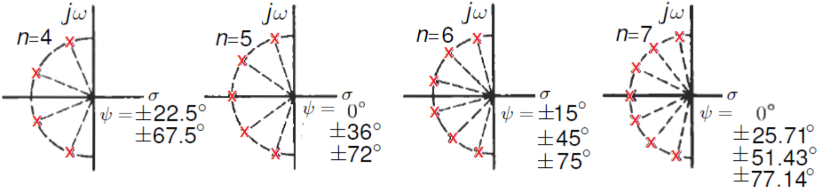
$$1 + S^4 = 0$$
$$(S + (0.707 \pm j0.707)) = 0$$
$$S^2 + \sqrt{2}S + 1$$

$n = 3$

$$1 - S^6 = 0$$
$$(S + 1)(S + \frac{1}{2} \pm j\frac{\sqrt{3}}{2}) = 0$$
$$(S + 1)(S^2 + S + 1)\psi =$$

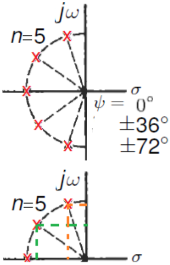


poles properties



- No poles on imaginary axis.
- If n is odd, then there is pole on the 0 degree.
- Poles are separated by $\psi = \dfrac{180^\circ}{n}$.
- $B_n = \left(\dfrac{1}{(S+1)} \right) \times \prod_k (S^2 + 2S\cos\psi_k + 1)$

pole location Example



$$B_n = \left(\frac{1}{(S+1)} \right) \times \prod_k (S^2 + 2S\cos\psi_k + 1)$$

$$\text{for } n = 5 \Longrightarrow \psi = \frac{180}{5} = 36^\circ$$

$$B_5 = (S+1)(S^2 + (2\cos 36)S + 1)(S^2 + (2\cos 72)S + 1)$$

$$B_5 = (S+1)(S^2 + (1.618)S + 1)(S^2 + (.618)S + 1)$$

$$B_5 = S^5 + 3.236S^4 + 5.236S^3 + 5.236S^2 + 3.236S + 1$$

$$Pole_0 = -1$$

$$Pole_{1,2} = cos(36) \pm sin(36) = 0.809 \pm j0.587$$

$$Pole_{3,4} = cos(72) \pm sin(72) = 0.309 \pm j0.951$$

Butterworth Poles Tables

TABLE 6.1 Pole Locations for Butterworth Responses

$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
−0.7071068	−0.5000000	−0.3826834	−0.8090170	−0.2588190
±j0.7071068	±j0.8660254	±j0.9238795	±j0.5877852	±j0.9659258
	−1.0000000	−0.9238795	−0.3090170	−0.7071068
		±j0.3826834	±j0.9510565	±j0.7071068
			−1.0000000	−0.9659258
				±j0.2588190

TABLE 6.2 Coefficients of the Butterworth Polynomial $B_n(s) = s^n + \sum_{i=0}^{n-1} a_i s^i$

n	a_0	a_1	a_2	a_3	a_4	a_5
2	1.0000000	1.4142136				
3	1.0000000	2.0000000	2.0000000			
4	1.0000000	2.6131259	3.4142136	2.6131259		
5	1.0000000	3.2360680	5.2360680	5.2360680	3.2360680	
6	1.0000000	3.8637033	7.4641016	9.1416202	7.4641016	3.8637033

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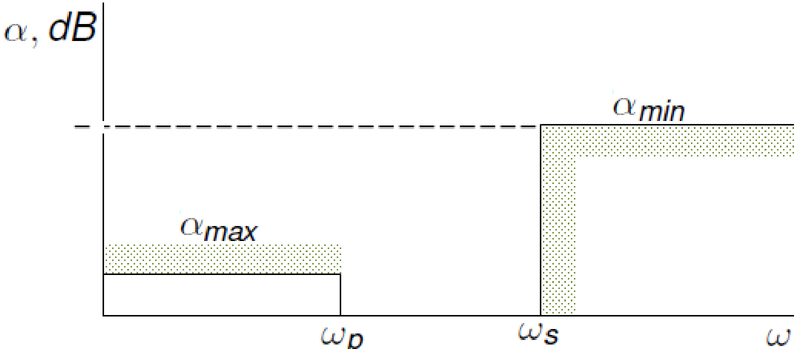
Filter order



$$n = \frac{\log\left(\frac{(10^{\frac{\alpha_{min}}{10}}-1)}{(10^{\frac{\alpha_{max}}{10}}-1)}\right)}{2\log\frac{\omega_S}{\omega_P}}$$

Filter order

- Filter Order

$$n = \frac{\log\left(\frac{(10^{\frac{\alpha_{min}}{10}}-1)}{(10^{\frac{\alpha_{max}}{10}}-1)}\right)}{2\log\frac{\omega_S}{\omega_P}} \tag{1}$$

- Calculate maximum attenuation (ε)

$$\varepsilon = \sqrt{(10^{\frac{\alpha_{max}}{10}} - 1)} \tag{2}$$

- Calculate Butterworth denormalized frequency (ω_B)

$$\omega_B = \varepsilon^{\frac{-1}{n}}\omega_p \tag{3}$$

Find pole locations, Q and gain

- Find Pole locations

$$B_n = \left(\frac{1}{(S+1)} \right) \times \prod_k (S^2 + 2S\cos\psi_k + 1) \qquad (4)$$

- Write the transfer function as $[\omega_0 = 1]$

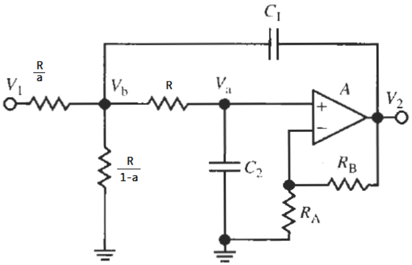
$$T(S) = \frac{K_1}{S+1} \frac{K_2}{S^2 + \frac{1}{Q}S + 1} \cdots \frac{K_n}{S^2 + \frac{1}{Q}S + 1} \qquad (5)$$

- Calculate the gain of each part (K)

$$k = 3 - \frac{1}{Q} \qquad (6)$$

K>1 or K<1

- K>1 insert amplifier at the end of your circuit
- K<1 change the first R to be resistor divider as next:



R and C

- Calculate R and C by $\omega_B = \frac{1}{RC}$
- Calculate R_B, R_A by $K = 1 + \frac{R_B}{R_A}$