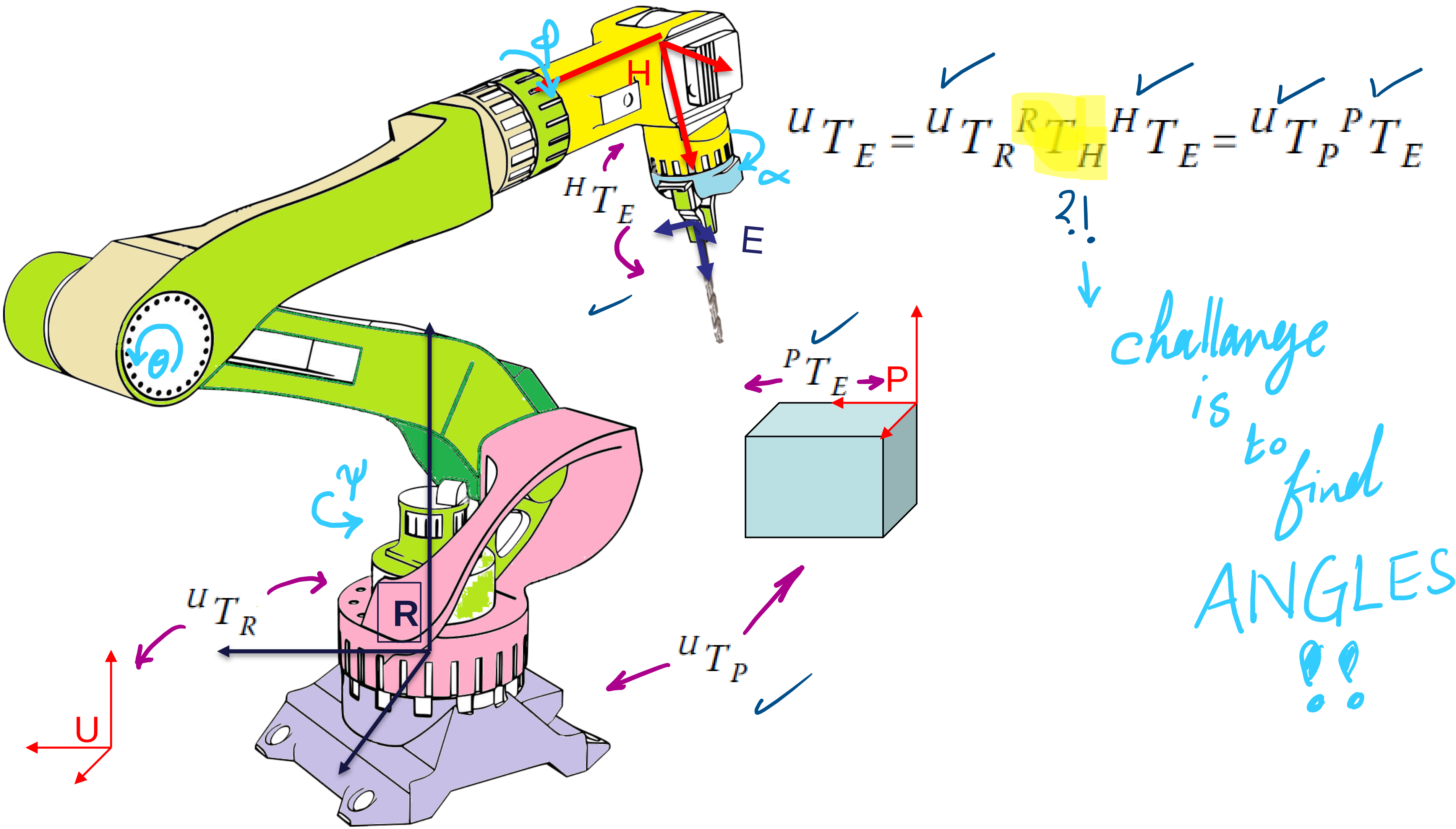
ROBOTICS KINEMATICS

Inverse Kinematics

By Mustafa Shiple

Inverse of Transformation



Inverse of Transformation

$$^{U}T_{E} = {}^{U}T_{R}\,{}^{R}T_{H}\,{}^{H}T_{E} = {}^{U}T_{P}\,{}^{P}T_{E}$$

$$[{}^{U}T_{R}]^{-1}\,[{}^{U}T_{R}\,{}^{R}T_{H}\,{}^{H}T_{E}]\,[{}^{H}T_{E}]^{-1} = [{}^{U}T_{R}]^{-1}\,[{}^{U}T_{P}\,{}^{P}T_{E}]\,[{}^{H}T_{E}]^{-1}$$

$$^{R}T_{H} = {}^{U}T_{R}^{-1}\,{}^{U}T_{P}\,{}^{P}T_{E}\,{}^{H}T_{E}^{-1}$$

$$= {}^{R}T_{U}\,{}^{U}T_{P}\,{}^{P}T_{E}\,{}^{E}T_{H}$$

Inverse matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

Pythagorean Trigonometric

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

Orthogonal matrices

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}^{-1} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} cos(\alpha) & -sin(\alpha) \\ sin(\alpha) & cos(\alpha) \end{bmatrix}^{-1} = \frac{1}{c^2(\alpha) + s^2(\alpha)} \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} = {G_{R_L}}^T$$

$${G_{R_L}}^{-1} = {G_{R_L}}^T$$ Orthogonal matrix

Quick Math Review (Dot Product)

$$= \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix} = {G_{R_L}}^T$$

$$Local: \quad \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \uparrow$$

Summary

$$Global: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\gamma) & -sin(\gamma) \\ 0 & sin(\gamma) & cos(\gamma) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

$$Local: \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\gamma) & sin(\gamma) \\ 0 & -sin(\gamma) & cos(\gamma) \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

$$Global: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} cos(\beta) & 0 & sin(\beta) \\ 0 & 1 & 0 \\ -sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

$$Local: \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} cos(\beta) & 0 & -sin(\beta) \\ 0 & 1 & 0 \\ sin(\beta) & 0 & cos(\beta) \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

$$Global: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & -sin(\alpha) & 0 \\ sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

$$Local: \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} cos(\alpha) & sin(\alpha) & 0 \\ -sin(\alpha) & cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix}$$

Properties of a Rotation Matrix

- Example 1

Do the following matrices represent rotation matrices?

a)
$$R_1 = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

b)
$$R_2 = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Solution

a) $\det R_1 = \frac{1}{2}$ ⇨ It is not a rotation matrix

b) $\left.\begin{aligned} \det R_2 &= 1 \\ R_2 R_2^T &= I \end{aligned}\right\}$ ⇨ It is a rotation matrix

4x4 matrix

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\mathbf{p}\cdot\mathbf{n} \\ o_x & o_y & o_z & -\mathbf{p}\cdot\mathbf{o} \\ a_x & a_y & a_z & -\mathbf{p}\cdot\mathbf{a} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example:

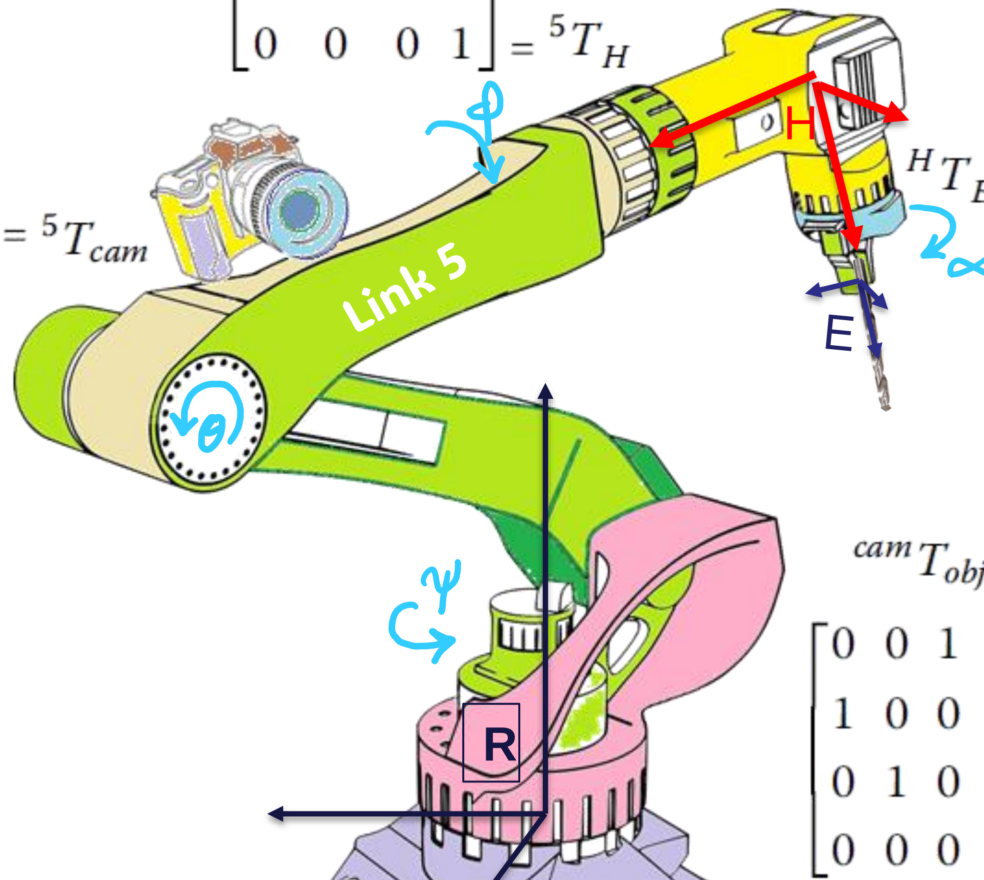
$$T = \begin{bmatrix} 0.5 & 0 & 0.866 & 3 \\ 0.866 & 0 & -0.5 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -(3\times 0.5 + 2\times 0.866 + 5\times 0) \\ 0 & 0 & 1 & -(3\times 0 + 2\times 0 + 5\times 1) \\ 0.866 & -0.5 & 0 & -(3\times 0.866 + 2\times -0.5 + 5\times 0) \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.866 & 0 & -3.23 \\ 0 & 0 & 1 & -5 \\ 0.866 & -0.5 & 0 & -1.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

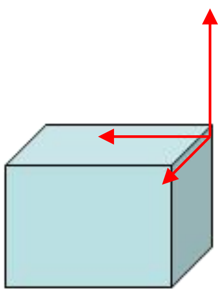
$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{5}T_{H}$$

$$\begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{5}T_{cam}$$



$${}^{H}T_{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{cam}T_{obj} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$${}^{R}T_{5} \times {}^{5}T_{H} \times {}^{H}T_{E} \times {}^{E}T_{obj} = {}^{R}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

solution

$${}^{R}T_{5} \times {}^{5}T_{H} \times {}^{H}T_{E} \times {}^{E}T_{obj} = {}^{R}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

$${}^{5}T_{H} \times {}^{H}T_{E} \times {}^{E}T_{obj} = {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

$${}^{H}T_{E} \times {}^{E}T_{obj} = {}^{5}T_{H}^{-1} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

$${}^{E}T_{obj} = {}^{H}T_{E}^{-1} \times {}^{5}T_{H}^{-1} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

$${}^{E}T_{obj} = {}^{E}T_{H} \times {}^{H}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

$${}^{5}T_{cam} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{5}T_{H} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{cam}T_{obj} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{H}T_{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

solution

$$ {}^{E}T_{obj} = {}^{E}T_{H} \times {}^{H}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj} $$

$$ {}^{5}T_{cam} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{5}T_{H} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{cam}T_{obj} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{H}T_{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{E}T_{H} = {}^{H}T_{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{H}T_{5} = {}^{5}T_{H}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

solution

$$^{E}T_{obj} = {}^{E}T_{H} \times {}^{H}T_{5} \times {}^{5}T_{cam} \times {}^{cam}T_{obj}$$

$${}^{5}T_{cam} = \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{H}T_{5} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{cam}T_{obj} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{E}T_{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{E}T_{obj} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DENAVIT-HARTENBERG REPRESENTATION

- The Denavit-Hartenberg (D-H) method of representation is a very simple way of modeling robot links and joints of any configuration, regardless of the sequence or complexity.
- the direct modeling of robots with the previous techniques is faster and more straight forward.
- D-H representation has an added benefit; analyses of differential motions and Jacobians, dynamic analysis, force analysis, and others are based on the results obtained from D-H representation

Prof.Dick Hartenberg

1950

Introduction

- Notation: Denavit - Hartenberg = *DH*
- It describes forward kinematics using 4 parameter for each joint: θ_i, d_i, a_i, α_i

Artic. i	d_i	θ_i	a_i	α_i
1	450	$180+q_1$	-150	90
2	0	$90+q_2$	600	0
3	0	$180+q_3$	-200	90
4	640	$180+q_4$	0	90
5	0	$180+q_5$	0	90
6	0	q_6	0	0

D-H Procedure

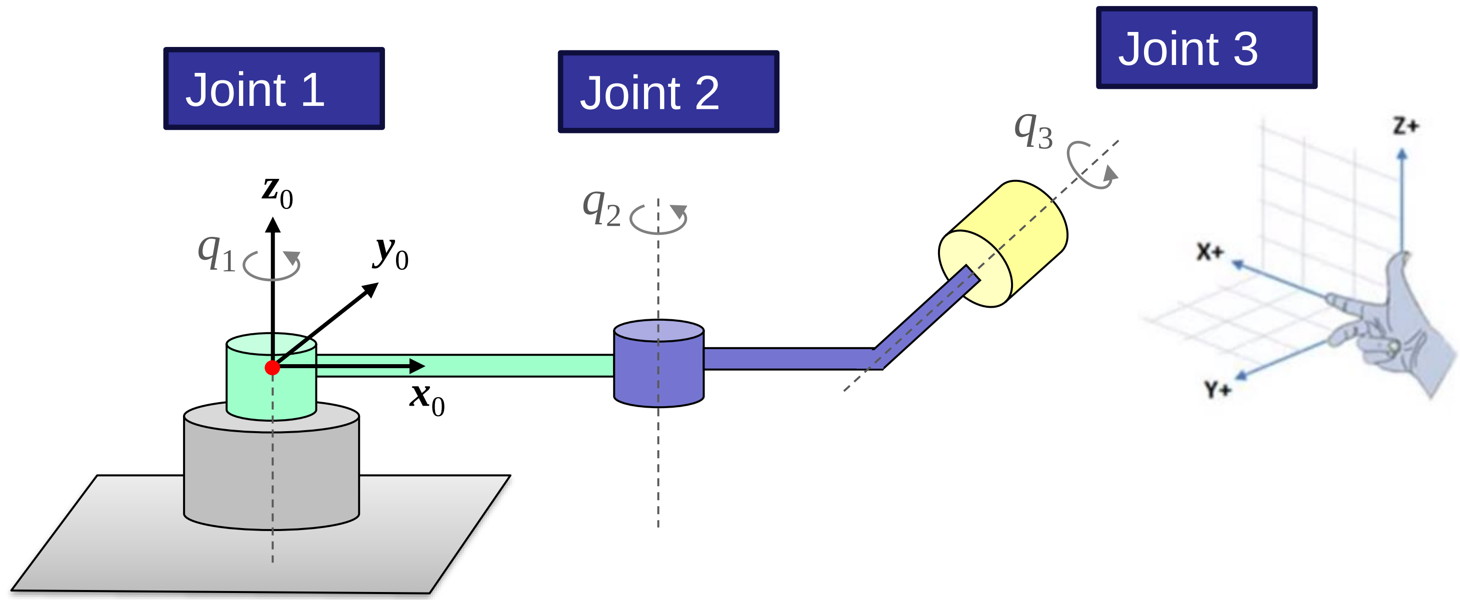
- It is a systematic (and classical) method used to describe forward kinematics of manipulators

- **Procedure**

1. Determine 1 frame per joint (based on some rules)
2. Determine 4 parameters (θ_i, d_i, a_i, α_i) that describe the pose between every two reference frames (based on rules)
3. Using the 4 parameters (per joint) compute the homogeneous transformation matrices

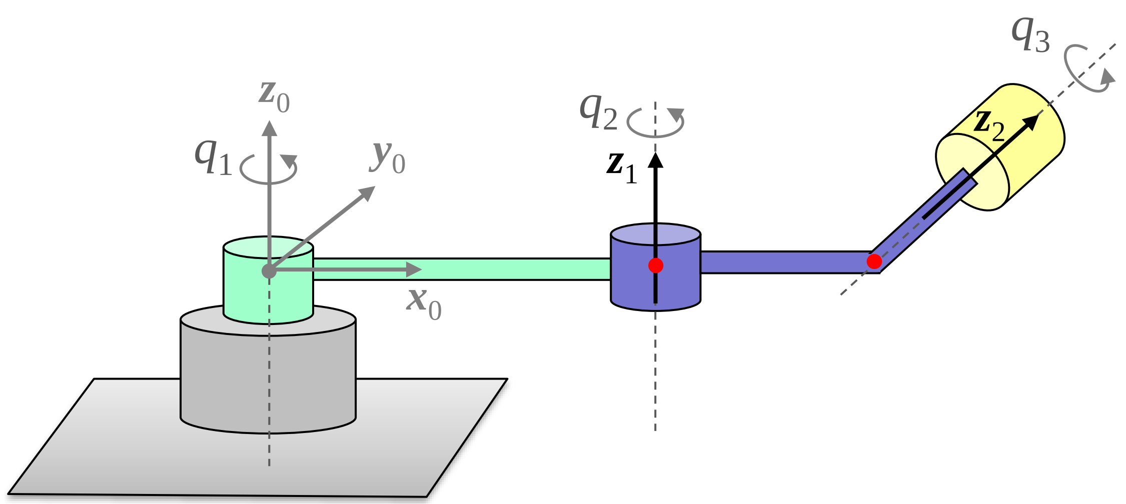
   → Determine the pose of the end effector with respect to the base (product of the homogeneous transformation matrices)

Step 1: Determine 1 frame per joint (based on some rules)



A. *Axis z_i*: align z_i with the axis of motion of joint $i+1$, If the joint is prismatic, the z axis is along the direction of the linear movement.

B. *Origin of frame $\{i\}$*: At the intersection of z_i & $z_{i\text{-}1}$. Or any point if they are in parallel

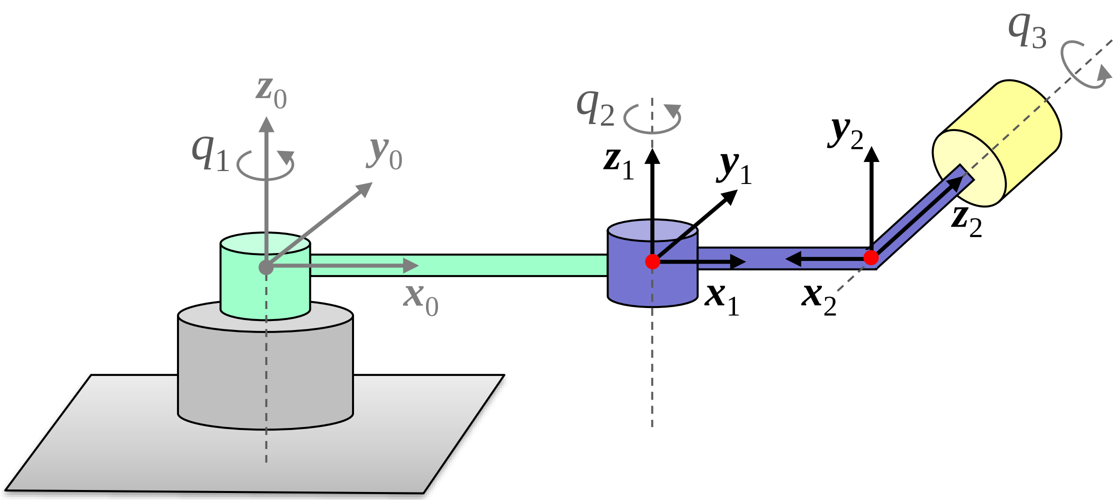


Denavit-Hartenberg Convention

Step 1: Determine 1 frame per joint (based on some rules)

C. *Axis x_i*: assign x_i in the direction of z_{i-1} x z_i. If (z_{i-1} & z_i) are parallel, assign x_i along the common normal between z_{i-1} & z_i

D. *Axis y_i*: assign y_i to complete the frame (following the right-hand rule)

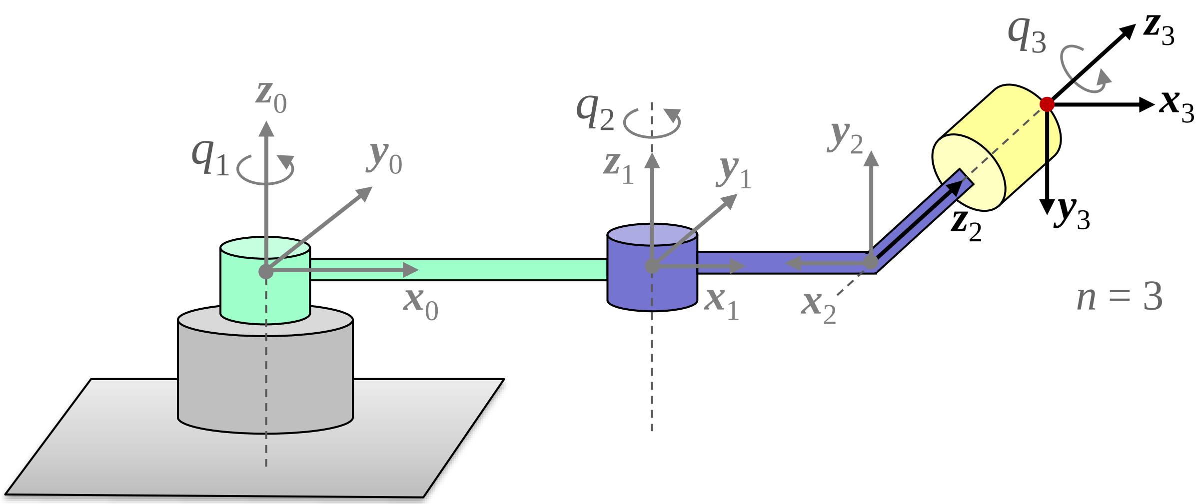


Denavit-Hartenberg Convention

Step 1: Determine 1 frame per joint (based on some rules)

D. End effector frame $\{n\}$:

- $\boldsymbol{x}_n$ must be orthogonal to $\boldsymbol{z}_{n\text{-}1}$ and it must intersect it (the origin of the frame is usually at the end of the kinematic chain)
- Usually $\boldsymbol{z}_n$ goes in the same direction as $\boldsymbol{z}_{n\text{-}1}$ pointing outwards
- $\boldsymbol{y}_n$ completes the frame (right hand rule)



Denavit-Hartenberg Convention

(2) Assigning the DH Parameters

<u>Joint Parameters</u>

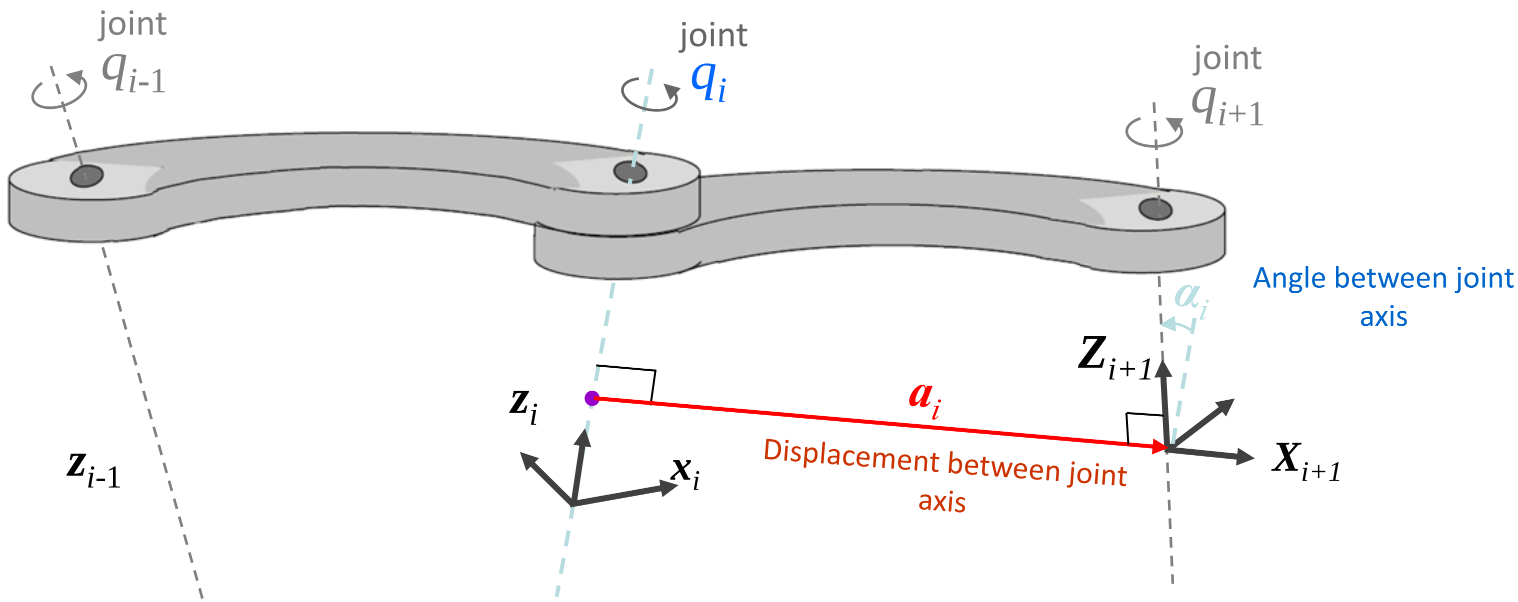
- *Joint angle* (θ_i)*:* rotation angle from axis $\boldsymbol{x}_{i\text{-}1}$ to axis $\boldsymbol{x}_i$ about axis $\boldsymbol{z}_{i\text{-}1}$
  - $\longrightarrow$ It is the joint variable if the i-th joint is revolute
- *Joint displacement* (d_i)*:* distance from the origin of frame $\{i\text{-}1\}$ to the intersection of axis $\boldsymbol{z}_{i\text{-}1}$ to axis $\boldsymbol{x}_i$ along axis $\boldsymbol{z}_{i\text{-}1}$
  - $\longrightarrow$ It is the joint variable if the i-th joint is prismatic

<u>Link Parameters</u> (constants)

- *Link length* (a_i)*:* distance from the intersection of axis $\boldsymbol{z}_{i\text{-}1}$ and axis $\boldsymbol{x}_i$ to the origin of frame $\{i\}$ along axis $\boldsymbol{x}_i$ *(shortest path)*
- *Link rotation angle* (α_i)*:* rotation angle from axis $\boldsymbol{z}_{i\text{-}1}$ to axis $\boldsymbol{z}_i$ about axis $\boldsymbol{x}_i$

Denavit-Hartenberg Convention

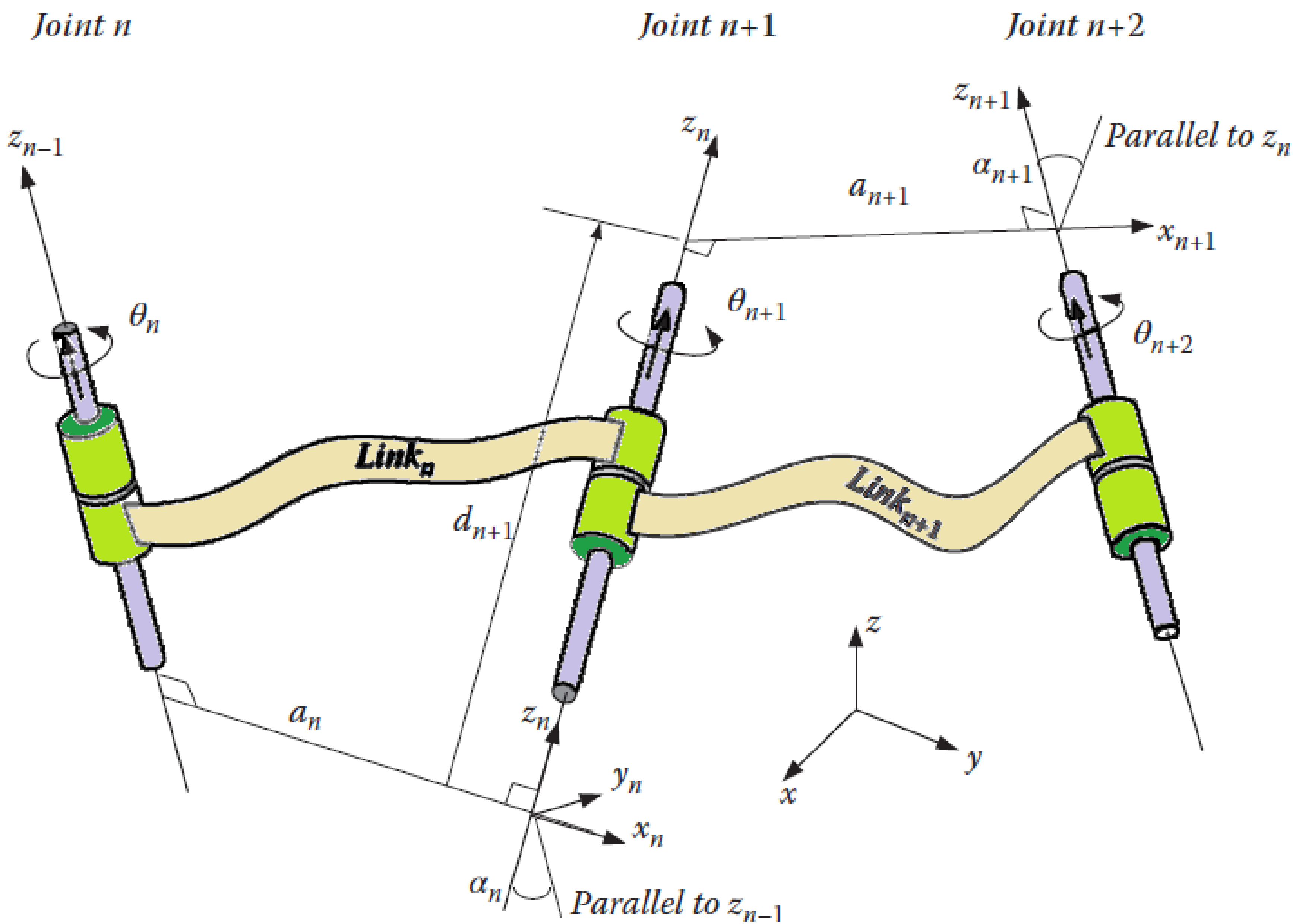
(2) Assigning the DH Parameters



Link parameters

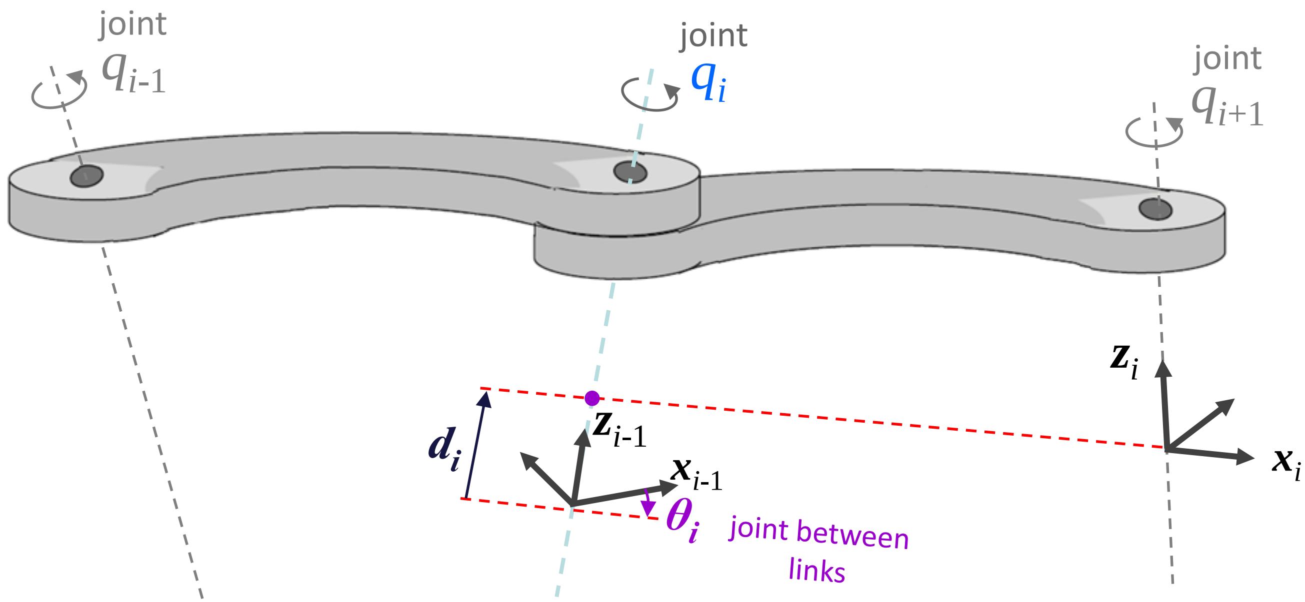
- $\boldsymbol{a_i}$: distance from [$\boldsymbol{z}_{i+1}$] to [$\boldsymbol{z}_i$] .by drawing a line perpendicular to both z axes
- α_i: angle from $\boldsymbol{z}_{i+1}$ to $\boldsymbol{z}_i$ about $\boldsymbol{x}_i$

Note: a_i, α_i have sign (they can be + or -)


Joint n
Joint n+1
Joint n+2
z_{n-1}
z_n
z_{n+1}
Parallel to z_n
α_{n+1}
a_{n+1}
x_{n+1}
θ_n
θ_{n+1}
θ_{n+2}
$Link_n$
$Link_{n+1}$
d_{n+1}
a_n
z_n
y_n
x_n
α_n
Parallel to z_{n-1}
z
y
x

Denavit-Hartenberg Convention

(2) Assigning the DH Parameters



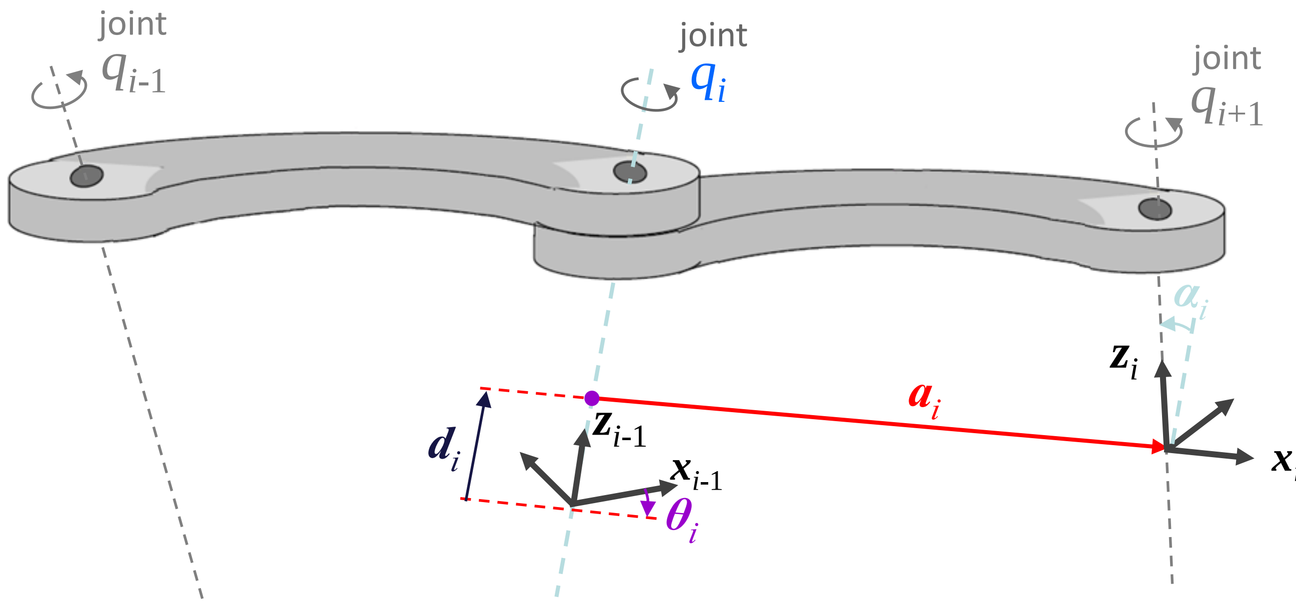
Joint parameters

- d_i: distance from the origin of $\{i\text{-}1\}$ to the [intersection of z_{i-1} with x_i] along z_{i-1}
- θ_i: rotation angle from x_{i-1} to x_i about z_{i-1}

Note: d_i, θ_i have sign (they can be + or -)

Denavit-Hartenberg Convention

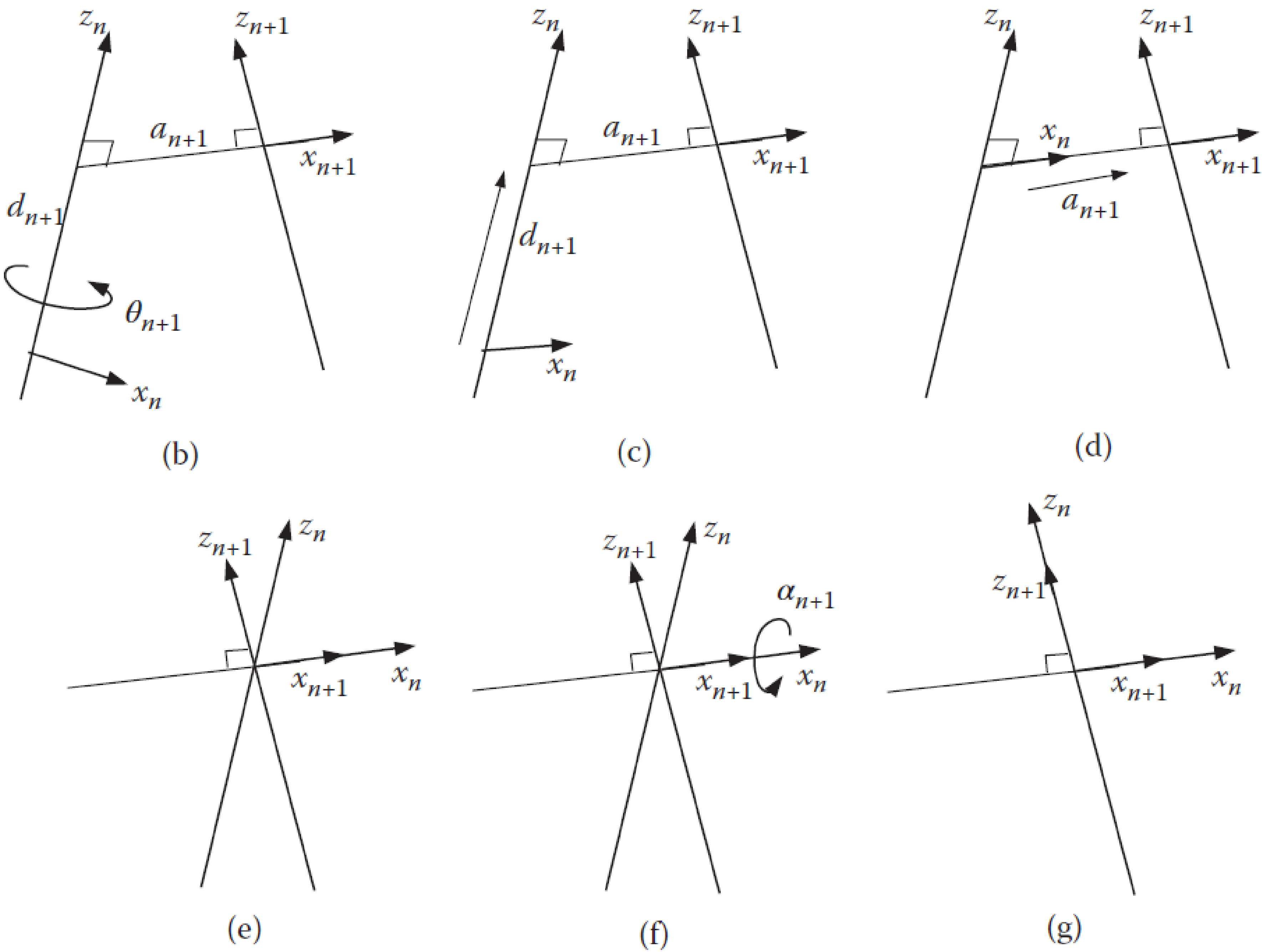
(2) Assigning the DH Parameters



Summary

- d_i: distance from the origin of $\{i\text{-}1\}$ to the [intersection of z_{i-1} with x_i] along z_{i-1}
- θ_i: rotation angle from x_{i-1} to x_i about z_{i-1}
- a_i: distance from [the intersection of z_{i-1} with x_i] to the origin of $\{i\}$ along x_i
- α_i: angle from z_{i-1} to z_i about x_i

[Denavit-Hartenberg Reference Frame Layout - YouTube]



z_n
z_{n+1}
a_{n+1}
x_{n+1}
d_{n+1}
θ_{n+1}
x_n
(b)
z_n
z_{n+1}
a_{n+1}
x_{n+1}
d_{n+1}
x_n
(c)
z_n
z_{n+1}
x_n
x_{n+1}
a_{n+1}
(d)
z_{n+1}
z_n
x_{n+1}
x_n
(e)
z_{n+1}
z_n
α_{n+1}
x_{n+1}
x_n
(f)
z_n
z_{n+1}
x_{n+1}
x_n
(g)

transformation $^{n}T_{n+1}$

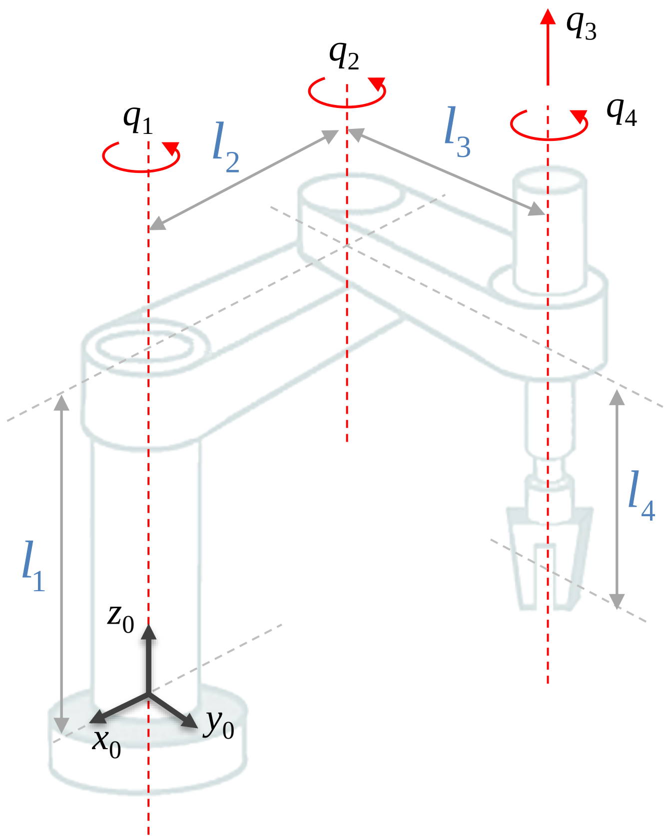
$$^{n}T_{n+1} = A_{n+1} = Rot(z,\theta_{n+1}) \times Trans(0,0,d_{n+1}) \times Trans(a_{n+1},0,0) \times Rot(x,\alpha_{n+1})$$

$$= \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1} & 0 & 0 \\ S\theta_{n+1} & C\theta_{n+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & a_{n+1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{n+1} & -S\alpha_{n+1} & 0 \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{n+1} = \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.53)$$

Example 1: DH of a SCARA Robot

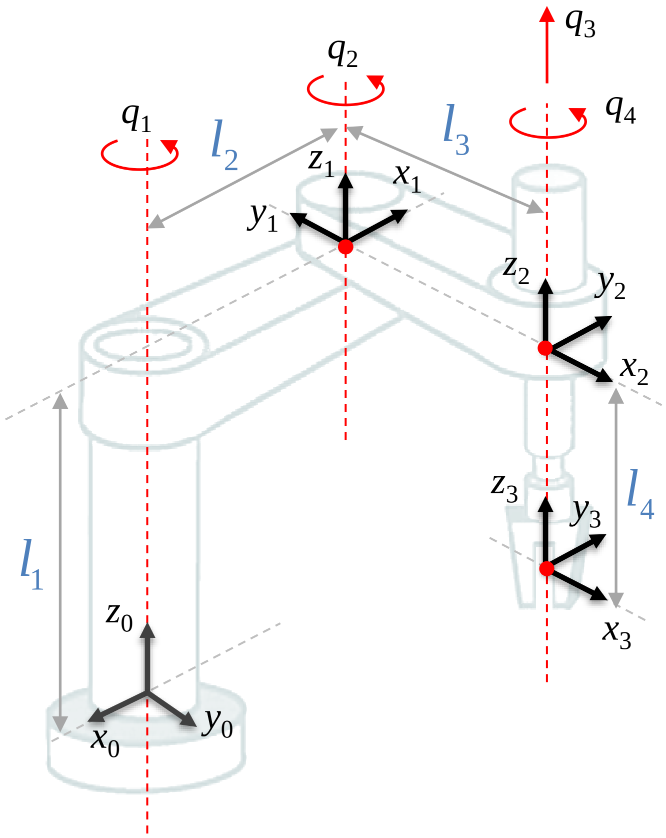
1. Reference frames



- Number joint axis
- Base reference frame: $\boldsymbol{z}_0$ along the axis of joint 1 (arbitrary origin, arbitrary x_0)

Example 1: DH of a SCARA Robot

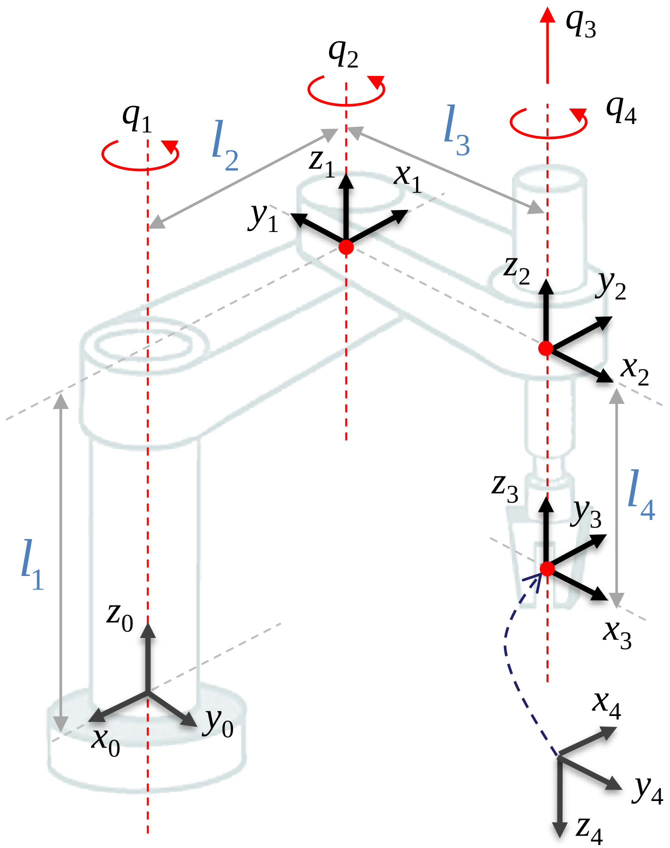
1. Reference frames



- Axis $\boldsymbol{z}_i$: $\boldsymbol{z}_i$ along the axis of joint i+1
- Origin of frame $\{i\}$:
  - a) Intersection of $\boldsymbol{z}_i$ & $\boldsymbol{z}_{i\text{-}1}$, or
  - b) Intersection of $\boldsymbol{z}_i$ with normal between $\boldsymbol{z}_i$ & $\boldsymbol{z}_{i\text{-}1}$ (If $\boldsymbol{z}_i$ & $\boldsymbol{z}_{i\text{-}1}$ parallel: arbitrary normal)
- Axis $\boldsymbol{x}_i$: in the direction of $\boldsymbol{z}_{i\text{-}1}$ x $\boldsymbol{z}_i$. If ($\boldsymbol{z}_{i\text{-}1}$ & $\boldsymbol{z}_i$) are parallel, $\boldsymbol{x}_i$ along their common normal
- Axis $\boldsymbol{y}_i$: assign $\boldsymbol{y}_i$ to complete the frame (using the right hand rule)

Example 1: DH of a SCARA Robot

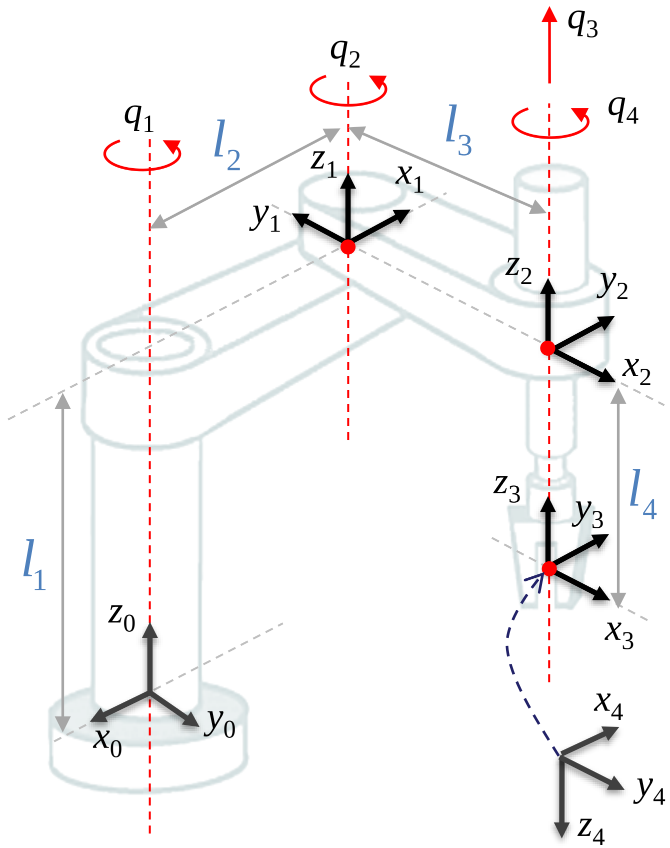
1. Reference frames



- End effector frame $\{n\}$:
  - $\boldsymbol{x}_n$ orthogonal to $\boldsymbol{z}_{n-1}$, intersecting it (origin at the end of the chain)
  - $\boldsymbol{z}_n$ in the direction of $\boldsymbol{z}_{n-1}$ pointing outwards
  - $\boldsymbol{y}_n$ completes the frame

Example 1: DH of a SCARA Robot

2. DH parameters



Joint *i*	d_i	θ_i	a_i	α_i
1	l_1	$180+q_1$	l_2	0
2	0	$-90+q_2$	l_3	0
3	$-l_4+q_3$	0	0	0
4	0	$90+q_4$	0	180

d_i: distance from $\{i\text{-}1\}$ to [intersection of z_{i-1} with x_i] along z_{i-1}
θ_i: angle from x_{i-1} to x_i alrededor de z_{i-1}
a_i: distance from [intersection of z_{i-1} wtih x_i] to $\{i\}$ along x_i
α_i: angle from z_{i-1} to z_i about x_i

Example 1: DH of a SCARA Robot

3. Homogeneous Transformation Matrices

Joint i	d_i	θ_i	a_i	α_i
1	l_1	$180+q_1$	l_2	0
2	0	$-90+q_2$	l_3	0
3	$-l_4+q_3$	0	0	0
4	0	$90+q_4$	0	180

$$^0T_1(q_1) = \begin{bmatrix} -\cos q_1 & \sin q_1 & 0 & -l_2\cos q_1 \\ -\sin q_1 & -\cos q_1 & 0 & -l_2\sin q_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3(q_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_3 - l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2(q_2) = \begin{bmatrix} \sin q_2 & \cos q_2 & 0 & l_3\sin q_2 \\ -\cos q_2 & \sin q_2 & 0 & -l_3\cos q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^3T_4(q_4) = \begin{bmatrix} -\sin q_4 & \cos q_4 & 0 & 0 \\ \cos q_4 & \sin q_4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Example 1: DH of a SCARA Robot

3. Homogeneous Transformation Matrices

- End effector with respect to the base:

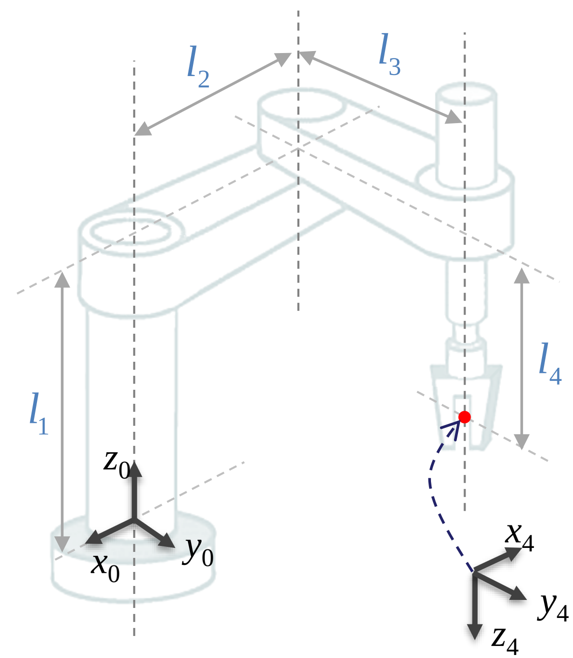
$$^0T_4 = \left(^0T_1\right)\left(^1T_2\right)\left(^2T_3\right)\left(^3T_4\right)$$

$$= \begin{bmatrix} -c_{124} & -s_{124} & 0 & -l_3 s_{12} - l_2 c_1 \\ -s_{124} & c_{124} & 0 & l_3 c_{12} - l_2 s_1 \\ 0 & 0 & -1 & l_1 - l_4 + q_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

- For the initial configuration ($q_1 = q_2 = q_3 = q_4 = 0$):

$$^0T_4 = \begin{bmatrix} -1 & 0 & 0 & -l_2 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & -1 & l_1 - l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Verify by inspection, in the diagram



Robot in the initial configuration

Compare with the result obtained using the geometric method

THE END